Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=y+z$. Then the equation becomes

$$y^7+z^7+(7yz+p)[y^5+z^5+(3yz+\tfrac{2}{7}p)(y^3+z^3)$$
$$+(5y^2z^2+\tfrac{5}{7}pyz+\tfrac{1}{49}p^2)(y+z)]+r=0.$$

Now x may be decomposed into two parts, y and z, in an infinite number of ways; we may, therefore, suppose y and z are such as to satisfy the condition $7yz+p=0$.

$\therefore\ yz=-\tfrac{1}{7}p,\ y^7+z^7=-r.$

Let $y^7=a$, $z^7=b$. $\therefore\ a+b=-r,\ ab=-(\tfrac{1}{7}p)^7$.

$\therefore\ a$ and b are the roots of the equation $u^2+ru-(\tfrac{1}{7}p)^7=0$.

$\therefore\ a=-\tfrac{1}{2}r+\sqrt{[\tfrac{1}{4}r^2+(\tfrac{1}{7}p)^7]},\ b=-\tfrac{1}{2}r-\sqrt{[\tfrac{1}{4}r^2+(\tfrac{1}{7}p)^7]}.$

Let ω be an imaginary seventh root of unity, so that

$$\omega=\tfrac{1}{2}[A\pm\sqrt{(A^2-4)}],\ A=\tfrac{1}{3}\sqrt[3]{\tfrac{7}{2}}[\sqrt[3]{(1+3\sqrt{-3})}+\sqrt[3]{(1-3\sqrt{-3})}-\sqrt[3]{\tfrac{2}{7}}].$$

Then the required seven roots of the equation are

$$\sqrt[7]{a}+\sqrt[7]{b},\quad \omega\sqrt[7]{a}+\omega^6\sqrt[7]{b},\quad \omega^2\sqrt[7]{a}+\omega^5\sqrt[7]{b},\quad \omega^3\sqrt[7]{a}+\omega^4\sqrt[7]{b},$$
$$\omega^6\sqrt[7]{a}+\omega\sqrt[7]{b},\quad \omega^5\sqrt[7]{a}+\omega^2\sqrt[7]{b},\quad \omega^4\sqrt[7]{\iota}+\omega^3\sqrt[7]{b}.$$

188. Proposed by GUY SCHUYLER.

$xy+ab=2ax,\quad x^2y^2+a^2b^2=2b^2y^2.$

Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

By squaring the first equation and subtracting from the second we get $y=xa/b$ and $y=-2xa/b$. Whence easily $x=b$, $\tfrac{1}{2}(-1\pm\sqrt{3})b$. $y=a(1\mp\sqrt{3})b/a$.

Also solved by G. W. Greenwood, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; G. W. Drake, Fayetteville, Ark.; Charles E. Barrett, Louisville, Ky.; H. F. MacNeish, A. B., Instructor in Mathematics in the University High School, Chicago, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va., and J. Scheffer, Kee Mar College, Hagerstown, Md.

GEOMETRY.

Problem 203 was also solved by Henry A. Converse, Ph. D., Instructor in Mathematics, Johns Hopkins University, Baltimore, Md.

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Tangents drawn to two confocal parabolae from the point on the common tangent intersect at the same angle as the axes of the parabolae.

I. Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

PQ is the common tangent; S, the common focus. Draw PA, PA' parallel to the axes; PT, PT' are the other tangents from P; T, T' being the points of tangency. Join PS.

Since a tangent from a point to a parabola makes the same angle with the line joining the point to the focus as the remaining tangent from the point does with the axis, we have $\angle TPA = \angle QPS = \angle T'PA'$. Hence $\angle TPT' = \angle APA'$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $y^2 = 4a(x+a)$.....(1),

$y^2\cos^2\theta + 2y\sin\theta(x\cos\theta + 2b) = 4bx\cos\theta + 4b^2 - x^2\sin^2\theta$.....(2),

be the confocal parabolae with axes inclined at an angle θ.

Let the common tangent be $y = mx + c$.....(3).

If (3) is tangent to (1), $c = a(m^2+1)/m$.

If (3) is tangent to (2), $c = b(m^2+1)/(m\cos\theta + \sin\theta)$.

$$\therefore\ m = \frac{a\cos\theta}{b - a\cos\theta},\quad c = \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)}.$$

$$\therefore\ (3)\text{ becomes } y = \frac{ax\sin\theta}{b - a\cos\theta} + \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)} \text{.....(4)}.$$

Let (h, k) be the point on (4).

$\therefore\ y = m_1x + a(m_1^2+1)/m_1$ is tangent to (1),

$y = m_2x + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$ is tangent to (2).

But $k = m_1h + a(m_1^2+1)/m_1$.....(5),

$k = m_2h + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$.....(6),

$$k = \frac{ah\sin\theta}{b - a\cos\theta} + \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)} \text{.....(7)}.$$

$$\text{From (5), } m_1^2 - \frac{km_1}{a+h} + \frac{a}{a+h} = 0 \text{.....(8)}.$$

$$\text{From (6), } m_2^2 + \frac{(h\sin\theta - k\cos\theta)m_2}{h\cos\theta + b} + \frac{b - k\sin\theta}{h\cos\theta + b} = 0 \text{.....(9)}.$$

$\therefore\ m_1$ and m_2 each has two values.

From (4), $m_1 = m_2 = a\sin\theta/(b - a\cos\theta)$. The other values of m_1 and m_2 are respectively,

$$\frac{b - a\cos\theta}{(a+h)\sin\theta},\quad \frac{(b - k\sin\theta)(b - a\cos\theta)}{a\sin\theta(h\cos\theta + b)} = \frac{b\cos\theta - h\sin^2\theta - a}{\sin\theta(h\cos\theta + b)}.$$

$$\therefore\ \frac{m_1 - m_2}{1 + m_1m_2} = \frac{(a^2 + b^2 - 2ab\cos\theta + 2ah\sin^2\theta + h^2\sin^2\theta)\sin\theta}{(a^2 + b^2 - 2ab\cos\theta + 2ah\sin^2\theta + h^2\sin^2\theta)\cos\theta} = \tan\theta.$$

209. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find by a geometrical method the maximum value of $\sin\theta\cos\theta\cos2\theta$.